QUANTUM GRAVITY ON A SQUARE GRAPH

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Abstract. We perform functional-integral quantisation of the moduli of all quantum metrics defined as square-lengths a on the edges of a Lorentzian quadrilateral graph. Noting that the partition function factorises into a theory for the spacelike edges and its conjugate for the timelike ones, we determine correlation functions and find a fixed relative uncertainty $\Delta a/\langle a\rangle = 1/\sqrt{8}$ for the edge square-lengths relative to their expected value $\langle a\rangle$. The expected value of the geometry is a rectangle where parallel edges have the same square-length. We compare with the simpler theory of a quantum scalar field on such a rectangular background. We also look at quantum metric fluctuations relative to a rectangular background, a theory which is finite and which at large rectangle scales resembles a pair of scalar fields on the vertical and horizontal edges with Planckian mass.

1. Introduction

The quantum spacetime hypothesis – the idea that the coordinates of spacetime are better modelled as noncommutative operators as an expression of quantum gravity effects was proposed in [31] on the basis of position-momentum reciprocity. The possibility itself was speculated on since the early days of quantum theory, while the specific argument in [31] was that the quantum phase space of some part of quantum gravity that contains position and momentum is obviously noncommutative, but its division into position and momentum is arbitrary and in particular should be interchangeable. Since there is generically gravity with curvature on spacetime then there should also generically be curvature in momentum space. But under quantum-group Fourier transform in the simplest cases, this is equivalent to noncommutative position space. Thus the search at the time for quantum groups that were both noncommutative and 'curved' (noncocommutative) became a toy model of the search for quantum gravity. The resulting 'bicrossproduct quantum groups' later resurfaced as quantum Poincaré groups for actual quantum spacetimes such as $[x_i, t] = \lambda x_i$, the Majid-Ruegg model [39] notable for its testable predictions via quantum Fourier transform[2]. Here the relevant quantum Poincaré group had been obtained in [28, 29] by contraction of $U_q(so_{3,2})$ while [39] identified it as a bicrossproduct and hence found the quantum spacetime on which it acts covariantly. Other early models were the Dopplicher-Fredenhagen-Roberts one [16] adapting work of Snyder[45] and the q-Minkowski one, see [32]. Quantum spacetimes and position-momentum duality are also visible in 3D quantum gravity[40] e.g. as a curved classical S^3 momentum space with the angular momentum algebra $U(su_2)$ as flat quantum spacetime[7]. Here $U(su_2)$ was first proposed as a noncommutative spacetime by t'Hooft[27] for other reasons.

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There have also been many works on quantum field theory on such flat quantum spacetimes. For the bicrossproduct model, modified Feynman rules due to the curved momentum space were already noted in [2] while interesting UV/IR mixing phenomena were found in [24]. For quantum field theory on $U(su_2)$ and related models, see notably [20, 21, 25] and references therein. How the $U(su_2)$ models relate to 3D loop or Chern Simons models was in [20, 21, 40, 42] among other works, and there are also links with loop quantum cosmology[3]. It was shown in recent work [37] that the 3D bicrossproduct model and the $U(su_2)$ model are in fact related by a Drinfeld twist and hence in some ways equivalent as flat quantum geometries.

To truly address issues of the unification of quantum theory and gravity we must, however, look beyond flat quantum spacetime models to models where spacetime is *both* curved and quantum. For this one needs a more sophisticated formalism and one that has emerged over the last two decades is a constructive approach to *quantum Riemannian geometry* [8, 9, 10, 12, 18, 19, 43, 30, 33, 38, 41] coming in part out of experience with the geometry of quantum groups but not limited to them. This starts with a quantum differential calculus and quantum metric, in contrast to the more well-known approach to noncommutative geometry of Connes starting with a 'Dirac operator' (spectral triple)[14], though not necessarily incompatible[10]. Recent lecture notes for the constructive formalism are in [34] and a brief outline is in Section 2.1. Our own motivation for this effort was that quantum Riemannian geometry should be a better-adapted starting point on which to build quantum gravity as it already includes the possibility of quantum gravity corrections. It is therefore a fair question as to whether, now, one can actually build models of quantum gravity using this more general conception of Riemannian geometry.

In this paper we carry such a model for the first time to completion, albeit a baby one with only four points forming a quadrilateral, for which the quantum Riemannian geometry was recently solved in [35, 13]. This uses the above formalism specialised to the case of directed graph, where the coordinate algebra is functions in the vertices, quantum differential forms are given by the directed edges and the metric is given by nonzero real numbers attached the latter. It is tempting to think of these 'edge weights' as lengths but in fact a better interpretation (e.g. from thinking about the graph Laplacian) is as the square of metric lengths. Note that there are potentially two such square-lengths for every edge, one for each direction, but we restrict to the 'edge-symmetric' case where these are the same. The formalism is recalled in Section 2.2 and Section 2.3 recalls the quadrilateral case, but adapted directly for a 'Lorentzian square graph' where horizontal edge weights will be taken negative. We will denote by square-length of an edge the magnitude of the number associated to an edge, and the geometric timelike or spacelike length is the square root of this. This is still an unquantised or 'classical' metric but using a more general 'quantum' notion of geometry to encode the discrete nature of the spacetime.

Section 3 starts the functional integral quantum theory with quantisation of a massive scalar free field on a Lorentzian rectangular background (where parallel edges have the same length and non-parallel edges are orthogonal) in Section 3.2. We also cover the scalar theory on a set of just two points and one edge in Section 3.1 and some results for a general curved non-rectangular background in Section 3.3. All of this should be seen as warm-up and we are not aiming to consider the quantisation of scalar fields on a lattice in any depth; for that see our complementary paper [36] about 1+0 dimensional scalar quantum theory on the lattice line, both the flat and the curved metric case. It may be possible to adapt aspects of the particle creation calculation on a curved background in [36] to the setting of Section 3.3, but we do not consider this here.

Section 4 then comes to our model of quantum gravity, starting with the full quantisation of the edge square-lengths where we integrate over all possible edge square-lengths of our Lorenzian quadrilateral. It is convenient to do this in momentum space where we write the two horizontal edge square-lengths equivalently as their average value $k_0 > 0$ (the coefficient of the zero mode) and their fluctuation k_1 (the coefficient of the nonzero momentum mode). Likewise for the time-like vertical edge lengths with average l_0 and fluctuation l_1. The resulting partition function Z on functionally integrating the natural Einstein-Hilbert action as a function of the edge square-lengths turns out to be real and positive and although divergent, we regularise the divergence by introducing an upper bound on the average lengths $k_0, l_0 < L$. We then obtain well-defined predictions by looking at relative correlation functions, such as a uniform relative uncertainty of $1/\sqrt{8}$ for all edge square-lengths as in the abstract. Unsurprisingly, integrating out the metric washes out much of the structure as the only remaining dependence is on the gravitational coupling constant G in the action, but this would nevertheless appear to be a first tangible result in this quantum gravity model.

Section 4.2 extracts more insight by writing

$$Z = \int_0^\infty \mathrm{d}k_0 \int_0^\infty \mathrm{d}l_0 \, Z(k_0, l_0)$$

where $Z(k_0, l_0)$ integrates only the k_1, l_1 non-zero momentum modes from our metric data but leaves the average values k_0, l_0 as background. This is inspired in part by the background field method in gauge theory and gravity where one integrates over all deviations from a fixed classical background, albeit our background is a Lorentzian rectangle rather a general quadrilateral. From this point of view the partition function for a fixed background becomes an effective action for the background field data. As such, this effective action behaves differently for large k_0, l_0 than for small k_0, l_0, namely

$$Z(k_0, l_0) \sim \begin{cases} \frac{\pi G}{2}\sqrt{k_0 l_0} & k_0, l_0 \to \infty \\ 4k_0 l_0 & k_0, l_0 \to 0 \end{cases}$$

where the sizes of k_0, l_0 are relative to the gravitational constant G, i.e. the first is really the $G \to 0$ or weak-gravity limit and turns out to essentially describe the product of two massless scalar fields on $\mathbb{Z}_2$ as in Section 3.1, with k_0, l_0 entering as coupling constants, namely a horizontal theory and a vertical theory. The other limit should be thought of as ‘deep quantum gravity’ since effectively $G \to \infty$ and this behaves very much *unlike* a pair of scalar fields and more like the full quantisation. We show that in this limit we do indeed find similar results for the relative uncertainties as in the full quantisation, which simply means that small k_0, l_0 modes dominate the full quantisation.

The paper concludes with some further directions that could be explored for this and similar models, as well as a general outlook. Finite quantum gravity has been considered in the past in Connes spectral triple approach, notably [26], but without directly comparable results due to the different starting point. Our approach also has a different character and methodology from current lattice quantum gravity.

We focus on the physical case of $\imath$ in the action, albeit the Euclideanised case could also be of interest, and we use units where $\hbar = c = 1$ and we retain one real coupling constant β in the action for the scalar case and another G for the metric case. In fact, standard dimension-counting does not apply in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model which is in some sense 0-dimensional (4 points) and in another sense 2-dimensional (there is a top form of degree 2 and a 2-dimensional cotangent bundle).

2. Preliminaries: formalism of quantum Riemannian geometry

We recall briefly some elements of the constructive approach to quantum Riemannian geometry as used particularly in [8, 9, 10, 12, 33, 38, 41], leading up to the Einstein-Hilbert and scaler field actions (2.17), (2.20) that we will need later. An introduction to the general formalism is [34] while the theory behind the discrete case that we specifically need is in [33]. The worked calculations for the $\mathbb{Z}_2$ example in Section 2.2 are new, while the formulae for the quadrilateral in Section 2.3 are mainly from [35] adapted to our Lorentzian case. The formalism is moreover the same as recently used for the integer line graph in our paper [36] and we will use the same notations as there.

2.1. **Bimodule approach.** We will not need the full generality of the theory and give only the bare bones at this general level, for orientation. It is important, however, that it exists so that our discrete geometry will be part of a functorial construction and not ad-hoc to the extent possible. Very explicit formulae at this level but in terms of bases and structure constants can be found in [38] if the reader prefers that.

In fact, it is well-known that classical geometry can be formulated equivalently in terms of a suitable algebra of functions on the space. The idea now is to allow this to be any algebra A with identity as our starting point (and now no actual space need exist). We replace the notion of differential structure on a space by specifying a bimodule Ω^1 of differential forms over A. A bimodule means we can multiply a '1-form' $\omega \in \Omega^1$ by 'functions' $a, b \in A$ either from the left or the right and the two should associate according to

$$(a\omega)b = a(\omega b). \tag{2.1}$$

We also need $\mathrm{d} : A \to \Omega^1$ an 'exterior derivative' obeying reasonable axioms, the most important of which is the Leibniz rule

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a(\mathrm{d}b) \tag{2.2}$$

for all $a, b \in A$. We usually require Ω^1 to extend to forms of higher degree to give a graded algebra $\Omega = \oplus\Omega^i$ (where associativity extends the bimodule identity (2.1) to higher degree). We also require d to extend to $\mathrm{d} : \Omega^i \to \Omega^{i+1}$ obeying a graded-Leibniz rule with respect to the graded product $\wedge$ and $\mathrm{d}^2 = 0$. This much structure is common to most forms of noncommutative geometry, including [14] albeit there it is not a starting point. In our constructive approach this 'differential structure' is the first choice we have to make in model building once we fixed the algebra A. We require that Ω is then generated by $A, \mathrm{d}A$ as it would be classically.

Next, on an algebra with differential we define a metric as an element $g \in \Omega^1 \otimes_A \Omega^1$ which is invertible in the sense of a map $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ which commutes with the product by A from the left or right and inverts g in the sense

$$((\omega,\ ) \otimes \mathrm{id})g = \omega = (\mathrm{id} \otimes (\ , \omega))g \tag{2.3}$$

for all 1-forms ω. In the general theory one can require quantum symmetry in the form $\wedge(g) = 0$, where we consider the wedge product on 1-forms as a map $\wedge : \Omega^1 \otimes_A \Omega^1 \to A$ and apply this to g.

Finally, we need the notion of a connection. A left connection on Ω^1 is a linear map $\nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ obeying a left-Leibniz rule

$$\nabla(a\omega) = \mathrm{d}a \otimes \omega + a\nabla\omega \tag{2.4}$$

for all $a \in A, \omega \in \Omega^1$. This might seem mysterious but if we think of a map $X : \Omega^1 \to A$ that commutes with the right action by A as a 'vector field' then we can evaluate ∇ as a covariant derivative $\nabla_X = (X \otimes \mathrm{id})\nabla : \Omega^1 \to \Omega^1$ which classically is then a usual covariant derivative on Ω^1. There is a similar notion for a connection on a general 'vector bundle' expressed algebraically but we only need the Ω^1 case. Moreover, when we have both left and right actions of A forming a bimodule as we do here, we say that a left connection is a *bimodule connection*[18, 19, 43, 30, 8] if there also exists a bimodule map σ such that

$$\sigma : \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1, \quad \nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes_A \mathrm{d}a) \tag{2.5}$$

for all $a \in A, \omega \in \Omega^1$. The map σ if it exists is unique, so this is not additional data but a property that some connections have. The key thing is that bimodule connections extend automatically to tensor products as

$$\nabla(\omega \otimes_A \eta) = \nabla\omega \otimes_A \eta + (\sigma(\omega \otimes_A (\ )) \otimes_A \mathrm{id})\nabla\eta \tag{2.6}$$

for all $\omega, \eta \in \Omega^1$, so that metric compatibility now makes sense as $\nabla g = 0$. A connection is called QLC or 'quantum Levi-Civita' if it is metric compatible and the torsion also vanishes, which in our language amounts to $\wedge\nabla = \mathrm{d}$ as equality of maps $\Omega^1 \to \Omega^2$.

We also have a Riemannian curvature for any connection,

$$R_\nabla = (\mathrm{d} \otimes_A \mathrm{id} - \mathrm{id} \wedge \nabla)\nabla : \Omega^1 \to \Omega^2 \otimes_A \Omega^1, \tag{2.7}$$

where classically one would interior product the first factor against a pair of vector fields to get an operator on 1-forms. Ricci requires more data and the current state of the art (but probably not the only way) is to introduce a lifting bimodule map $i : \Omega^2 \to \Omega^1 \otimes_A \Omega^1$. Applying this to the left output of R_∇ we are then free to 'contract' by using the metric and inverse metric to define $\mathrm{Ricci} \in \Omega^1 \otimes_A \Omega^1$ [9]. The associated Ricci scalar and the geometric quantum Laplacian are

$$S = (\ ,\ )\mathrm{Ricci} \in A, \quad \Delta = (\ ,\ )\nabla\mathrm{d} : A \to A \tag{2.8}$$

defined again along lines that generalise these classical concepts to any algebra with differential structure, metric and connection.

Finally, and critical for physics, are unitarity or 'reality' properties. We work over $\mathbb{C}$ but assume that A is a $*$-algebra (real functions, classically, would be the self-adjoint elements). We require this to extend to Ω as a graded-anti-involution (so reversing order with an extra signs when odd degree differential forms are involved) and to commute with d. 'Reality' of the metric and of the connection in the sense of being $*$-preserving are imposed as [8, 9]

$$g^\dagger = g, \quad \nabla \circ * = \sigma \circ \dagger \circ \nabla; \quad (\omega \otimes_A \eta)^\dagger = \eta^* \otimes_A \omega^* \tag{2.9}$$

where $\dagger$ is the natural $*$-operation on $\Omega^1 \otimes_A \Omega^1$. These 'reality' conditions in a self-adjoint basis (if one exists) and in the classical case would ensure that the metric and connection coefficients are real.

2.2. **Quantum Riemannian geometry of a single edge.** We will be interested in the case of X a discrete set and $A = \mathbb{C}(X)$ the usual commutative algebra of complex functions on it as our 'spacetime algebra'. It can be shown (basically by considering the action of δ-functions) that for such an algebra the possible differential structures (Ω^1, d) are in 1-1 correspondence with directed graphs with X as the set of vertices, cf [14, 33, 34]. A directed graph just means to draw at most one arrow between some of the vertices, and no self-arrows are allowed. In fact for the calculus to admit a quantum metric the graph needs to be bidirected, i.e. whenever there is an arrow $x \to y$ there is also and arrow $y \to x$, in other words our data will just be an undirected graph where $x - y$ means an arrow in both directions. The reason this graph language is useful is that Ω^1 has a basis $\{\omega_{x\to y}\}$ over $\mathbb{C}$ exactly labelled by the arrows of the graph. We then define the bimodule structure and differential

$$f.\omega_{x\to y} = f(x)\omega_{x\to y},\quad \omega_{x\to y}.f = \omega_{x\to y}f(y),\quad \mathrm{d}f = \sum_{x\to y}(f(y) - f(x))\omega_{x\to y}$$

and in the bidirected case a quantum metric has the form [33]

$$g = \sum_{x\to y} g_{x\to y}\omega_{x\to y} \otimes_{\mathbb{C}(X)} \omega_{y\to x}$$

with weights $g_{x\to y} \in \mathbb{R}\setminus\{0\}$ for every arrow. The calculus over $\mathbb{C}$ is compatible with complex conjugation on functions $f^*(x) = \overline{f(x)}$ and $\omega^*_{x\to y} = -\omega_{y\to x}$, from which we see that 'reality' of the metric in (2.9) indeed amounts to real metric weights. It is not required mathematically, but reasonable from the point of view of the physical interpretation, to restrict attention to the *edge symmetric* case where $g_{x\to y} = g_{y\to x}$ is independent of the direction. Finding a QLC for a metric depends on how Ω^2 is defined and one choice is to just set $\Omega^2 = 0$ (thinking of the graph as 1-dimensional). Another choice is a kind universal or 'maximal prolongation' of Ω^1, but the choice of interest will typically be a quotient of this, for example to have Ω^2 1-dimensional as for a 2-manifold. The possible choices depend on the graph.

The most convenient special type of graph here is a Cayley graph. This is defined by a group structure on X and $\mathcal{C}$ a set of generators of X. We then define a graph by $x \to xi$ whenever $i \in \mathcal{C}$ (the product here is the group product). In other words we can 'step around' on X but only by right multiplying by a generator. This case is convenient because then there is a natural basis of left-invariant 1-forms $e_i = \sum_{x\to xi}\omega_{x\to xi}$. These obey the more algebraic rules

$$e_i f = R_i(f)e_i,\quad \mathrm{d}f = \sum_i(\partial^i f)e_i,\quad \partial^i = R_i - \mathrm{id},\quad R_i(f)(x) = f(xi) \tag{2.10}$$

defined by the right translation operators R_i as stated. Moreover, if $\mathcal{C}$ is a union of conjugacy classes then Ω is generated by the e_i with certain 'braided-anticommutation relations' cf. [46] and a certain form of $\mathrm{d}e_i$. In the case of an Abelian group this is just the usual Grassmann algebra on the e_i (they anticommute) and $\mathrm{d}e_i = 0$, and hence very easy to work with. As with the geometry of Lie groups, it is much easier to do calculations in a basis of left-invariant forms (or vector fields).

Since all of this may be unfamiliar to readers, we explain the above in complete detail in the case of the dumbbell graph $0 \leftrightarrow 1$ for a set $X = \{0, 1\}$ of two points. There are two arrows $\omega_{0\to1}$ and $\omega_{1\to0}$ so these are a basis over $\mathbb{C}$ of Ω^1. The differential of a function f is

$$\begin{aligned}
\mathrm{d}f &= (f(1)-f(0))\omega_{0\to 1} + (f(0)-f(1))\omega_{1\to 0}\\
&= ((f(1)-f(0))\delta_0 + (f(0)-f(1))\delta_1)(\omega_{0\to 1}+\omega_{1\to 0})\\
&= (\partial^1 f)e_1
\end{aligned}$$

where the first expression is the graph form but we then write the dumbbell as the Cayley graph for the group $X=\mathbb{Z}_2$ with the unique generator $\mathcal{C}=\{1\}$ and one invariant form with

$$e_1 = \omega_{0\to 1}+\omega_{1\to 0},\quad e_1 f = \tilde f e_1,\quad \mathrm{d}f = (\partial^1 f)e_1.$$

Here $\tilde f$ swaps the values at $0,1$ and the left invariant vector field ∂^1 dual to e_1 is

$$\partial^1 f = \tilde f - f,\quad (\partial^1 f)(0) = f(1)-f(0) = -(\partial^1 f)(1).$$

The reader can check that the left and right actions on $\omega_{0\to 1},\omega_{1\to 0}$ result in the commutation relations between e_1 and any function f as stated.

For the exterior algebra and $*$-structure we set $e_1^2=0$, $\mathrm{d}e_1=0$ and $e_1^*=-e_1$ and meanwhile the general form of metric is given by a real-valued function a and has the form

$$g = ae_1\otimes_{\mathbb{C}(\mathbb{Z}_2)} e_1 = a(0)\omega_{0\to 1}\otimes_{\mathbb{C}(\mathbb{Z}_2)}\omega_{1\to 0} + a(1)\omega_{1\to 0}\otimes_{\mathbb{C}(\mathbb{Z}_2)}\omega_{0\to 1}$$

with the corresponding two non-vanishing real edge-weights $a(0)=g_{0\to 1}$ and $a(1)=g_{1\to 0}$. This explains both a metric coefficient function a point of view and the graph edge square-lengths point of view on the metric, and how they are related. The edge-symmetric case where we have the same weight associated to either direction entails the restriction $a(1)=a(0)$ or $\partial^1 a = 0$. The inverse metric is $(e_1,e_1)=1/\tilde a$.

Lemma 2.1. *There exists a QLC for the above calculus and metric on $\mathbb{Z}_2$ if and only if $a(1)=\pm a(0)$, in which case the $*$-preserving QLCs take the form*

$$\nabla e_1 = be_1\otimes e_1,\quad b(0)=1-q,\quad b(1)=1-q^{-1}\rho;\quad \rho=\frac{a(1)}{a(0)}=\pm 1$$

for a free parameter q with $|q|=1$.

Proof. A connection has to have the form $\nabla e_1 = be_1\otimes e_1$ for some function b since e_1 is a basis over the algebra, and we then extend this as $\nabla(fe_1)=\mathrm{d}f\otimes e_1 + f\nabla e_1 = (\partial^1 f + fb)e_1\otimes e_1$ so as to obey the left Leibniz rule. $\Omega^2=0$ since $e_1^2=0$, so the connection is automatically torsion free. To be a bimodule connection we need $\nabla(\tilde f e_1)=\nabla(e_1 f) = (\nabla e_1)f + \sigma(e_1\otimes \mathrm{d}f)$ from which we deduce that if it exists then

$$(\partial^1\tilde f)\sigma(e_1\otimes e_1) = \sigma(e_1\otimes\partial^1 fe_1) = (\partial^1\tilde f + \tilde f b)e_1\otimes e_1 - fbe_1\otimes e_1 = (\partial^1\tilde f)(1-b)e_1\otimes e_1$$

using the commutation relations in the calculus and that σ is a bimodule map defined over $\otimes_{\mathbb{C}(X)}$ for the first equality. This has to be true for all f, so we deduce that

$$\sigma(e_1\otimes e_1) = (1-b)e_1\otimes e_1$$

which is indeed a well-defined bimodule map as it is consistent with the commutation relations of the calculus. Now we check metric compatibility as

$$\nabla g = \nabla(ae_1\otimes e_1) = (\partial^1 a + ab)e_1\otimes e_1\otimes e_1 + a\sigma(e_1\otimes be_1)\otimes e_1 = 0$$

which for the form of σ and the commutation rules to move the b inside to the left translates to $\partial^1 a + ab + a\tilde{b}(1-b) = 0$ or the two equations

$$\frac{a(1)}{a(0)} - 1 + b(0) + b(1) - b(0)b(1), \quad \frac{a(0)}{a(1)} - 1 + b(0) + b(1) - b(0)b(1).$$

These require $(a(1)/a(0))^2 = 1$ or $a(1) = \pm a(0)$. Assuming this, we can set $b(0)$ freely as $1-q$ for a parameter q and deduce $b(1)$ as stated. Finally, we need the connection to be 'real' in the sense of $*$-preserving as in (2.9) which is

$$-be_1 \otimes e_1 = -\nabla e_1 = \nabla e_1^* = \sigma \circ \dagger \circ \nabla e_1 = \sigma(e_1 \otimes e_1 b^*) = b^*(1-b)e_1 \otimes e_1$$

or $-b = b^*(1-b)$ which is $q - 1 = (1-q^*)q$ or $|q| = 1$ when evaluated at 0. It also then holds at the other point. Thus the moduli space of $*$-preserving QLCs is a circle parametrized by q. □

The QLCs found above necessarily have zero curvature as $\Omega^2 = 0$ and have geometric Laplacian

(2.11)
$$\Delta f = (\ ,\ )\nabla \mathrm{d} f = (\ ,\ )\nabla(\partial^1 f e_1) = (\ ,\ )(\partial^{1^2} f + (\partial^1 f)b)(e_1 \otimes e_1) = -(\partial^1 f)\left(\frac{q}{\tilde{a}} + \frac{q^{-1}}{a}\right).$$

We focus on the case $a(1) = a(0)$ case so that the metric is edge-symmetric (so there is just a single real number $a \in \mathbb{R}$ associated to the edge) and $\Delta f = -(\partial^1 f)(q + q^{-1})/a$ then has real as opposed to imaginary eigenvalues. In this case the scalar action for a free massive field is

$$S_f = \sum_{\mathbb{Z}_2} \mu f^*(\Delta + m^2) f = (q + q^{-1})|f(1) - f(0)|^2 + am^2\left(|f(0)|^2 + |f(1)|^2\right) \tag{2.12}$$

where we see that edge weight $a \in \mathbb{R}$ has the square of length dimension so that am^2 is dimensionless. Thinking of the $\mathbb{Z}_2$ in the time direction, we assumed $a > 0$ and used $\mu = a$ as the constant 'measure' in the sum. Although our derivation of (2.12) from the formalism was quite involved, our result for a complex scalar function f on two points is clear enough with the obvious kinetic and mass terms on $\mathbb{Z}_2$ as a 2-step lattice. The hidden freedom in the geometry from the choice of QLC just appearing as a real scale factor in front of the kinetic term. There is nothing stopping one adding an interaction term to (2.12), for example $\sum_{\mathbb{Z}_2} \mu f^3 = a(f(0)^3 + f(1)^3)$ in the case of real-valued field.

2.3. **Quantum Riemannian geometry of a quadrilateral.** We now consider $X = \mathbb{Z}_2 \times \mathbb{Z}_2$ with its canonical 2D calculus given by a quadrilateral with vertices $00, 01, 10, 11$ in an abbreviated notation as shown in Figure 1. We view the graph as a Cayley graph for the group as indicated and with generators $10, 01$. There are 8 arrows so Ω^1 is 8-dimensional over $\mathbb{C}$ but as we saw in detail for $\mathbb{Z}_2$, it is convenient to work with a basis over $\mathbb{C}(X)$ of left-invariant 1-forms, namely

$$e_1 = \omega_{00\to10} + \omega_{01\to11} + \omega_{10\to00} + \omega_{11\to01}, \quad e_2 = \omega_{00\to01} + \omega_{10\to11} + \omega_{01\to00} + \omega_{11\to10}.$$

with relations and derivative (2.10) now having the form $e_i f = (R_i f)e_i$ with translation invariant vector fields $\partial^i = R_i - \mathrm{id}$, where R_1 takes the other value of the first coordinate of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and R_2 takes the other value of the second. Explicitly,

$$(R_1 f)(i,j) = f(i+1, j), \quad (R_2 f)(i,j) = f(i, j+1) \tag{2.13}$$

$$(\partial^1 f)(i,j) = f(i+1,j) - f(i,j), \quad (\partial^2 f)(i,j) = f(i,j+1) - f(i,j) \tag{2.14}$$

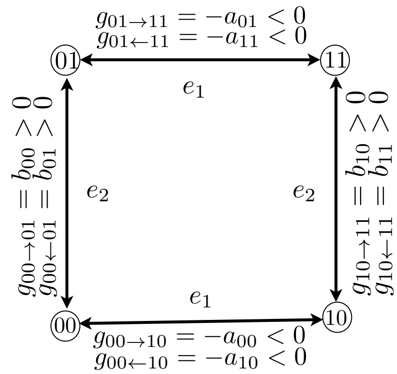

FIGURE 1. Metric coefficients defined by functions $a, b > 0$ are interpreted [35] as arrow square-lengths on a Lorentzian square graph with e_1 spacelike.

if we write coordinates $(i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ with entries taken mod 2. The $*$-exterior algebra is the usual Grassmann algebra on the e_i (they anticommute) with $e_i^* = -e_i$. The general form of a quantum metric and its inverse according to the general scheme above are

$$g = -ae_1 \otimes e_1 + be_2 \otimes e_2, \quad (e_1, e_1) = -\frac{1}{R_1 a}, \quad (e_2, e_2) = \frac{1}{R_2 b}, \quad (e_1, e_2) = (e_2, e_1) = 0$$

where the coefficients are real functions a, b with positive values to reflect a Lorentzian signature in which e_1 is the spacelike direction. In terms of the graph, their 8 values are equivalent to the $g_{x\to y}$ weights for the 8 arrows according to Figure 1, where $a_{ij} = a(i,j)$ is a shorthand and similarly for b_{ij}. As for the $\mathbb{Z}_2$ case above, it is natural to focus on the edge-symmetric case where the edge weight assigned to an edge does not depend on the direction of the arrow. In our case this now means

$$\partial^1 a = \partial^2 b = 0 \tag{2.15}$$

and we assume this in what follows. The Cayley graph method using invariant forms, while not essential, allows one to solve for the QLC by algebraic methods much as we did in detail for $\mathbb{Z}_2$ in Lemma 2.1, with the result given in the Euclidean generic metric case in [35]. As the method is the same, we omit further details and state the results in the form we need now.

Case 1: Generic metric QLCs. For generic (non-constant) metrics we adapt the QLCs found in [35] to the Lorentzian case with $a \to -a$ in present conventions. There is a 1-parameter family of QLCs

$$\nabla e_1 = (1 + Q^{-1})e_1 \otimes e_1 + (1 - \alpha)(e_1 \otimes e_2 + e_2 \otimes e_1) + \frac{b}{a}(R_2\beta - 1)e_2 \otimes e_2,$$

$$\nabla e_2 = \frac{a}{b}(R_1\alpha - 1)e_1 \otimes e_1 + (1 - \beta)(e_1 \otimes e_2 + e_2 \otimes e_1) + (1 - Q)e_2 \otimes e_2,$$

where Q, α, β are functions on the group defined as

$$Q = (q, q^{-1}, q^{-1}, q), \quad \alpha = (\frac{a_{01}}{a_{00}}, 1, 1, \frac{a_{00}}{a_{01}}), \quad \beta = (1, \frac{b_{10}}{b_{00}}, \frac{b_{00}}{b_{10}}, 1) \tag{2.16}$$

when we list the values on the points in the above vertex order. Here q is a free parameter and we need $|q|=1$ for a $*$-preserving connection.

The Riemann curvature has the general form $R_\nabla e_i = \rho_{ij} e_1\wedge e_2\otimes e_j$ where [35]

$$\rho_{11} = Q^{-1}R_1\alpha - Q\alpha + (1-\alpha)(R_1\beta-1) + \frac{R_2 a}{a}(R_2\beta-1)(R_2R_1\alpha-1)$$
$$\rho_{12} = Q^{-1}(1-\alpha)+\alpha(R_2\alpha-1) - Q^{-1}\frac{R_1 b}{a}(\beta^{-1}-1)) - \frac{b}{a}(R_2\beta-1)R_2\beta$$

and similar formulae for ρ_{2i}. If we use the obvious antisymmetric lift $i(e_1\wedge e_2) = \frac{1}{2}(e_1\otimes e_2 - e_2\otimes e_1)$ then

$$\mathrm{Ricci} = ((\ ,\ )\otimes\mathrm{id})(\mathrm{id}\otimes i\otimes\mathrm{id})(\mathrm{id}\otimes R_\nabla)(g) = \frac{1}{2}\begin{pmatrix} -R_2\rho_{21} & -R_2\rho_{22} \\ R_1\rho_{11} & R_1\rho_{12}\end{pmatrix}$$

as the matrix of coefficients on the left in our tensor product basis. Applying $(\ ,\ )$, the resulting Ricci scalar curvature is

$$S = \frac{1}{2}\left(\frac{R_2\rho_{21}}{a} + \frac{R_1\rho_{12}}{b}\right) = \frac{1}{4ab}\left((3+q-(1-q)\chi)\frac{\partial^2 a}{\alpha} + (1-q^{-1}-(3+q^{-1})\chi)\frac{\partial^1 b}{\beta}\right)$$

where $\chi=(1,-1,-1,1)$. Next, we take $\mu=ab>0$ in our conventions and

$$S_g = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2}\mu S = (a_{00}-a_{01})^2\left(\frac{1}{a_{00}}+\frac{1}{a_{01}}\right) - (b_{00}-b_{10})^2\left(\frac{1}{b_{00}}+\frac{1}{b_{10}}\right) \tag{2.17}$$

independently of q. If we had taken the Euclidean signature as in [35] then both terms would enter with + and the minimum would be zero, for the constant or rectangular case. If we want to add a cosmological term to the Einstein-Hilbert action, this means to change S to $S-\Lambda$ which adds to S_g above an extra term

$$-\Lambda\sum_{\mathbb{Z}_2\times\mathbb{Z}_2}\mu = -\Lambda(a_{00}+a_{01})(b_{00}+b_{10}). \tag{2.18}$$

We will not actually add this as it complicates the calculations and we are moreover interested in the idea of a vacuum energy appearing as a quantum geometry correction.

Finally, the geometric Laplacian for the generic metric solutions comes out as[35]

$$\Delta f = (\ ,\ )\nabla((\partial^i f)e_i) = -\left(\frac{Q^{-1}-R_2\beta}{a}\right)\partial^1 f - \left(\frac{Q+R_1\alpha}{b}\right)\partial^2 f \tag{2.19}$$

again with a change of sign for a. While the derivation of these formulae following the formalism is quite involved, the main thing we will need in the sequel is the resulting Einstein-Hilbert action (2.17) as a function of the metric coefficients (equivalent to the four independent real positive parameters $a_{00}, a_{01}, b_{00}, b_{10}$ for the edge-lengths as in Figure 1) and a scalar field action

$$S_f = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2}\mu f^*(\Delta+m^2)f \tag{2.20}$$

with measure $\mu=ab>0$ and Δ the Laplacian (2.19). Here R_i and ∂^i were given explicitly in (2.13)–(2.14) and the functions Q,α,β were given in (2.16).

Case 2: Constant metric QLCs. It is not central to the paper but we mention that in the 'rectangular' case where a, b are constant, so $\alpha = \beta = 1$, there is a much larger 4-parameter moduli of QLCs given in [13] by $P = (p_1, p_2^{-1}, p_1^{-1}, p_2)$ and $Q = (q_1, q_1^{-1}, q_2^{-1}, q_2)$ for nonzero constants p_i, q_i, where as before we list the values on the points in order $00, 01, 10, 11$. The connections and curvature take the form[13]

$$\nabla e_1 = (1 - P)e_1 \otimes e_1, \quad \nabla e_2 = (1 - Q)e_2 \otimes e_2,$$

$$R_\nabla e_1 = (\partial^2 P)e_1 \wedge e_2 \otimes e_1, \quad R_\nabla e_2 = -(\partial^1 Q)e_1 \wedge e_2 \otimes e_2,$$

with the connection $*$-preserving when $|p_i| = |q_i| = 1$. So the moduli space of QLCs here is the 4-torus T^4. The Ricci tensor for the same antisymmetric lift as before now gives

$$\mathrm{Ricci} = \frac{1}{2}\left((\partial^2 P^{-1})e_2 \otimes e_1 - (\partial^1 Q^{-1})e_1 \otimes e_2\right), \quad S = 0.$$

The Laplacian for the 4-parameter QLCs for constant a, b is

$$\Delta f = \left(\frac{P+1}{a}\right)\partial^1 f - \left(\frac{Q+1}{b}\right)\partial^2 f \tag{2.21}$$

again with change of sign of a compared to [13]. The moduli of QLCs for generic metrics in Case 1 reduces when the metric is constant to the special case $P = -Q^{-1}$ with $q_1 = q_2$. There may also in principle be intermediate QLCs when just one of a or b is non-constant.

3. Quantisation of free scalar fields on two and four points

We will adopt a 'functional integral' approach where we parameterise the fields and integrate an action over all fields. It is convenient to work in momentum space where our fields are Fourier transformed on the underlying finite group. Before doing this, we remind the reader how this works for the more familiar case of functions on $\mathbb{Z}_n$, namely we can expand a function $f \in \mathbb{C}(Z_n)$ as

$$f(i) = \sum_{j=0}^{n-1} f_j \phi_j(i) = \sum_{j=0}^{n-1} f_j e^{\frac{2\pi \imath i j}{n}}; \quad \phi_j(i) = e^{\frac{2\pi \imath i j}{n}}.$$

Here $\phi_0 = 1, \cdots, \phi_{n-1}$ are the different frequency plane waves on $\mathbb{Z}_n$ and the coefficient of the zero mode is the average value $f_0 = \frac{1}{n}\sum_i f(i)$ of the function, the coefficient of the fundamental mode is f_1, etc.

Now, for $\mathbb{Z}_2$ we have only two momentum modes $\phi_0 = 1$ and $\phi_1 = \phi$ the sign function defined by $\phi(i) = (-1)^i$, and Fourier mode expansion means to write

$$f = f_0 + f_1\phi; \quad f_0 = \frac{f(0) + f(1)}{2}, \quad f_1 = \frac{f(0) - f(1)}{2}$$

as inverse to $f(0) = f_0 + f_1$ and $f(1) = f_0 - f_1$. So working in momentum space on $\mathbb{Z}_2$ just means working with the average f_0 and the fluctuation f_1 instead of the original function values $f(0), f(1)$. All the benefits of working in momentum space still apply, notably finite difference operators become diagonalised in the f_i description.

3.1. **Scalar field on a single edge.** For simplicity, we take f real-valued (the complex case has the same form of free field action for the real and imaginary components separately). Then making the Fourier expansion as above, the action (2.12) for a scalar field immediately becomes

$$S_f = 4(q+q^{-1})f_1^2 + 2am^2(f_0^2+f_1^2).$$

The path integral $Z = 2\int \mathrm{d}f_0\mathrm{d}f_1 e^{\frac{\imath}{\beta}S_f}$ has a Gaussian form which we compute as usual using

$$Z_\alpha = \int_{-\infty}^{\infty} \mathrm{d}k\, e^{\imath\alpha k^2} = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\imath\pi}{4}}$$

which implies

$$\langle k^2\rangle := \frac{\int_{-\infty}^{\infty}\mathrm{d}k\, e^{\imath\alpha k^2}k^2}{\int_{-\infty}^{\infty}\mathrm{d}k\, e^{\imath\alpha k^2}} = \frac{1}{\imath Z_\alpha}\frac{\partial}{\partial\alpha}Z_\alpha = \frac{\imath}{2\alpha} \tag{3.1}$$

and hence in our case the correlation functions

$$\langle f(0)f(1)\rangle = \langle f(1)f(0)\rangle = \langle f_0^2 - f_1^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{am^2} - \frac{1}{am^2+2(q+q^{-1})}\right) \tag{3.2}$$

$$\langle f(0)f(0)\rangle = \langle f(1)f(1)\rangle = \langle f_0^2 + f_1^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{am^2} + \frac{1}{am^2+2(q+q^{-1})}\right) \tag{3.3}$$

where $\langle f_0f_1\rangle = 0$ as each integrand is then odd. There is a divergence as $m\to 0$ but in the massive case there are no divergences and hence no renormalisation needed until we consider interactions. This is an IR divergence in the sense that it appears as $m\to 0$ but note that a is the metric coefficient or square-length and only the dimensionless combination am^2 enters, so the divergence more precisely appears when the Compton wavelength tends to infinity relative the edge length. We recall that q is a dimensionless phase and reflects a freedom in the underlying quantum geometry. It can be absorbed into the choice of constant β and a change of mass, albeit it could change the sign and hence the signature. If we take β dimensionless then f should be dimensionless, which will be the case of interest later. But if we wanted f to have $\sqrt{\text{length}}$ dimension as usual in 1 dimensions then we would need β to be a length scale.

3.2. **Scalar fields on a Lorentzian rectangle.** We start with the constant metric case so $a,b>0$ are constant horizontal and vertical edge square-lengths (with the former entering as a negative edge weight), and we work in 'momentum space' by expanding in terms of the four Fourier modes

$$1,\quad \phi(i,j)=(-1)^i=(1,1,-1,-1),\quad \psi(i,j)=(-1)^j=(1,-1,1,-1),\quad \phi\psi=\chi$$
$$\partial^1\phi=-2\phi,\quad \partial^2\phi=0,\quad \partial^1\psi=0,\quad \partial^2\psi=-2\psi,\quad \partial^1\chi=\partial^2\chi=-2\chi.$$

Thus, we let

$$f = f_0 + f_1\phi + f_2\psi + f_3\chi$$

for the plane wave expansion of a general scalar field. As before, we focus on the real-valued case so the f_i are real. We are mainly interested in this paper in generic metrics so we start with the specialisation of the generic metric Laplacian (2.19) to the case a,b constant of interest now. Then $\alpha=\beta=1$ in (2.16) but we still have a circle parameter q for the QLC and we write the corresponding functions $Q^{\pm1}$ as

$$Q=\frac{1}{2}\left(q+q^{-1}+(q-q^{-1})\chi\right),\quad Q^{-1}=\frac{1}{2}\left(q+q^{-1}-(q-q^{-1})\chi\right).$$

Then

$$\Delta f = 2\frac{Q^{-1}-1}{a}(f_1\phi + f_3\chi) + 2\frac{Q+1}{b}(f_2\psi + f_3\chi)$$

$$\Delta 1 = 0, \quad \Delta\phi = \frac{q+q^{-1}-2}{a}\phi - \frac{q-q^{-1}}{a}\psi, \quad \Delta\psi = \frac{q+q^{-1}+2}{b}\psi + \frac{q-q^{-1}}{b}\phi$$

$$\Delta\chi = (q-q^{-1})\left(-\frac{1}{a}+\frac{1}{b}\right) + \left(\frac{q+q^{-1}-2}{a} + \frac{q+q^{-1}+2}{b}\right)\chi$$

has one zero mode 1, one mode built from $1,\chi$ with real eigenvalue read off from the bottom line and two more modes which are linear combinations of ϕ,ψ and which one can show also have real eigenvalues for $a,b>0$. Then the scalar field action in (2.20) in our flat case becomes

$$S_f = 4\Big(a\frac{(q+1)^2}{q}(f_2^2+f_3^2) + b\frac{(q-1)^2}{q}(f_1^2+f_3^2) + (q-q^{-1})(a-b)(f_1f_2+f_0f_3) \\ + m^2ab\left(f_0^2+f_1^2+f_2^2+f_3^2\right)\Big).$$

The action again has real coefficients if and only if $q=\pm1$ so that the $q-q^{-1}$ term vanishes. This is also the case where this class of QLCs with the rectangular metric has zero quantum Riemann curvature, $R_\nabla = 0$. We now focus on this case where the interpretation is clearer. Then

$$S_f = \begin{cases} 16a(f_2^2+f_3^2) & \text{if } q=1 \\ -16b(f_1^2+f_3^2) & \text{if } q=-1 \end{cases} + 4m^2ab\left(f_0^2+f_1^2+f_2^2+f_3^2\right) \tag{3.4}$$

This is now in diagonal form so we can immediately write down the functional integral quantisation using

$$Z = \int \mathrm{d}f_0\mathrm{d}f_1\mathrm{d}f_2\mathrm{d}f_3 e^{\frac{\imath}{\beta}S_f}$$

and regarding the a,b,m as parameters. If the coupling constant β is of square length dimension then f should be dimensionless as for a usual scalar theory in 2 dimensions. All four integrals have the same Gaussian form as the $\mathbb{Z}_2$ case, so from (3.1) we can compute 2-point functions for $q=1$ as

$$\begin{aligned}
\langle f_{00}f_{01}\rangle &= \langle f_{10}f_{11}\rangle = \langle f_0^2+f_1^2-f_2^2-f_3^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{abm^2} - \frac{1}{abm^2+4a}\right)\\
\langle f_{00}f_{10}\rangle &= \langle f_{01}f_{11}\rangle = \langle f_0^2-f_1^2+f_2^2-f_3^2\rangle = 0\\
\langle f_{00}f_{11}\rangle &= \langle f_{01}f_{10}\rangle = \langle f_0^2-f_1^2-f_2^2+f_3^2\rangle = 0\\
\langle f_{ij}^2\rangle &= \langle f_0^2+f_3^2+f_1^2+f_2^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{abm^2} + \frac{1}{abm^2+4a}\right)
\end{aligned}$$

where we use the shorthand $f_{ij} = f(i,j)$ so that $f_{00} = f_0+f_1+f_2+f_3$, $f_{01} = f_0+f_1-f_2-f_3$ and $f_{00}f_{01} = (f_0+f_1)^2-(f_2+f_3)^2$, etc. As before, the cross terms do not contribute due to the parity of the integrands. The result for $q=-1$ is similar,

$$\begin{aligned}
\langle f_{00}f_{01}\rangle &= \langle f_{10}f_{11}\rangle = \langle f_0^2+f_1^2-f_2^2-f_3^2\rangle = 0\\
\langle f_{00}f_{10}\rangle &= \langle f_{01}f_{11}\rangle = \langle f_0^2-f_1^2+f_2^2-f_3^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{abm^2} - \frac{1}{abm^2-4b}\right)\\
\langle f_{00}f_{11}\rangle &= \langle f_{01}f_{10}\rangle = \langle f_0^2-f_1^2-f_2^2+f_3^2\rangle = 0\\
\langle f_{ij}^2\rangle &= \langle f_0^2+f_3^2+f_1^2+f_2^2\rangle = \frac{\imath\beta}{4}\left(\frac{1}{abm^2} + \frac{1}{abm^2-4b}\right).
\end{aligned}$$

Finally, because we are in the rectangular metric case, the quantum Riemannian geometry actually admits a larger moduli of QLCs with Laplacian (2.21) where

$$\Delta f = -2\frac{P+1}{a}(f_1\phi + f_3\chi) + 2\frac{Q+1}{b}(f_2\phi + f_3\chi).$$

The P, Q depend on four modulus 1 parameters p_i, q_i and a similar analysis to the above gives the action has real coefficients if and only if p_i, q_i have values ± 1 or P, Q are chosen from $\pm 1, \pm\chi$. For example, $P = Q = 1$ has

$$\Delta f = \frac{2}{a}\partial^1 f - \frac{2}{b}\partial^2 f = -\frac{4}{a}(f_1\phi + f_3\chi) + \frac{4}{b}(f_2\phi + f_3\chi).$$

Hence $\Delta 1 = 0$, $\Delta\phi = -\frac{4}{a}\phi$, $\Delta\psi = \frac{4}{b}\psi$ and $\Delta\chi = (-\frac{4}{a} + \frac{4}{b})\chi$ gives us the eigenmodes modes, with just one zero mode. We also have action

$$S_f = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu f(\Delta + m^2) f = -16b(f_1^2 + f_3^2) + 16a(f_2^2 + f_3^2) + 4abm^2(f_0^2 + f_1^2 + f_2^2 + f_3^2)$$

for a massive free field again with 'measure' $\mu = ab$. This again has diagonal form which is a composite of our previous $q = \pm 1$ cases. Then we can immediately write down the 2-point functions as

$$\begin{aligned}
\langle f_{00} f_{01}\rangle = \langle f_{10} f_{11}\rangle &= \frac{\imath\beta}{8}\left(\frac{1}{abm^2} + \frac{1}{abm^2 - 4b} - \frac{1}{abm^2 + 4a} - \frac{1}{abm^2 + 4a - 4b}\right)\\
\langle f_{00} f_{10}\rangle = \langle f_{01} f_{11}\rangle &= \frac{\imath\beta}{8}\left(\frac{1}{abm^2} - \frac{1}{abm^2 - 4b} + \frac{1}{abm^2 + 4a} - \frac{1}{abm^2 + 4a - 4b}\right)\\
\langle f_{00} f_{11}\rangle = \langle f_{01} f_{10}\rangle &= \frac{\imath\beta}{8}\left(\frac{1}{abm^2} - \frac{1}{abm^2 - 4b} - \frac{1}{abm^2 + 4a} + \frac{1}{abm^2 + 4a - 4b}\right)\\
\langle f_{ij}^2\rangle &= \frac{\imath\beta}{8}\left(\frac{1}{abm^2} + \frac{1}{abm^2 - 4b} + \frac{1}{abm^2 + 4a} + \frac{1}{abm^2 + 4a - 4b}\right)
\end{aligned}$$

where $f_{i,j} = f(i,j)$. As before, the massless case of the above would have an infra-red divergence, here regularised by the mass parameter m.

3.3. **Scalar field on a curved non-rectangular background.** Here we briefly consider the general case of a scalar field with a general (generically curved) non-rectangular edge-symmetric metric. In this case, it is convenient to also Fourier expand the metric in terms of four real momentum-space coefficients as

$$a = k_0 + k_1\psi, \quad b = l_0 + l_1\phi \tag{3.5}$$

$$a_{00} = k_0 + k_1, \quad a_{01} = k_0 - k_1, \quad b_{00} = l_0 + l_1, \quad b_{10} = l_0 - l_1 \tag{3.6}$$

where in each case only two Fourier modes enter because of the restructions $\partial^1 a = \partial^2 b = 0$ for edge symmetry. The preceding section was further restricted to $k_1 = l_1 = 0$ and $a = k_0, b = l_0$ while more generally $k_0, l_0 > 0$ are each the average of two parallel edge square-lengths (with the actual horizontal metric edge weights being negative) and k_1, l_1 are the amount of fluctuation. We restrict to $|k_1| < k_0$ and $|l_1| < l_0$ in order that our metric does not change signature. In either case it is useful to change variables from k_1, l_1 to the *relative fluctuations*

$$k = \frac{k_1}{k_0}, \quad l = \frac{l_1}{l_0} \tag{3.7}$$

both in the interval $(-1, 1)$. As before, we keep the scalar field real valued for simplicity (the complex case is entirely similar).

We need the 1-parameter QLCs for the generic metric, with a modulus one parameter q, Laplacian (2.19) and resulting in scalar field action (2.20), which in momentum terms now comes out as

$$\begin{aligned}S_f = 4\Big(& k_0\frac{(q+1)^2}{q}(f_2^2+f_3^2)+l_0\frac{(q-1)^2}{q}(f_1^2+f_3^2)\\ &+(q+q^{-1})\,(kk_0(f_0f_2+f_1f_3)+ll_0(f_0f_1+f_2f_3))\\ &+(q-q^{-1})(k_0-l_0)(f_1f_2+f_0f_3)-2ll_0\left(q-q^{-1}-2\right)f_1f_3+2kk_0\left(q-q^{-1}+2\right)f_2f_3\\ &+m^2k_0l_0\left(f_0^2+f_1^2+f_2^2+f_3^2+2l(f_0f_1+f_2f_3)+2k(f_0f_2+f_1f_3)+2kl(f_1f_2+f_0f_3)\right)\Big)\end{aligned}$$

As a check, this is invariant under the interchange

$$q\ \leftrightarrow\ -q^{-1};\quad k_0\ \leftrightarrow\ -l_0;\quad k\ \leftrightarrow\ l;\quad f_1\ \leftrightarrow\ f_2. \tag{3.8}$$

In the Euclidean square graph version (before we changed a to $-a$) we have $k_0\leftrightarrow l_0$ and the symmetry reflects the ability to interchange the horizontal and vertical directions of the square, but note that we also have to change q. A similar symmetry was noted for the eigenvalues of Δ in [35] but the above is more relevant since it accounts also for the 'measure' μ in the action.

On the other hand, we again need $q=\pm1$ for the action to have real coefficients (to kill the $q-q^{-1}$) term) and, without loss of generality, we focus on the case $q=1$; the other case is similar given the symmetry mentioned above. In this case

$$\begin{aligned}S_f = 4\Big(&4k_0(f_2^2+f_3^2)+2kk_0(f_0f_2+f_1f_3)+2ll_0(f_0f_1+f_2f_3)+4ll_0f_1f_3+4kk_0f_2f_3\\ &+m^2k_0l_0\left(f_0^2+f_1^2+f_2^2+f_3^2+2l(f_0f_1+f_2f_3)+2k(f_0f_2+f_1f_3)+2kl(f_1f_2+f_0f_3)\right)\Big).\end{aligned}$$

Moreover, the action is quadratic in the f_i so the functional integration is that of a Gaussian, with the result that the partition function for a free field can be treated in just the same way as the in the case of a rectangular background in Section 3.2, after diagonalisation of the quadratic form underlying S_f. At issue for this are the eigenvalues of this quadratic form. Its trace is

$$8(k_0\frac{(q+1)^2}{q}+l_0\frac{(q-1)^2}{q}+2m^2k_0l_0)=16k_0(2+m^2l_0)$$

when $q=1$. Thus the sum of the eigenvalues (even for complex q) is real but the eigenvalues themselves for generic values are complex unless $q=\pm1$, when they are real. They are also generically but not necessarily nonzero (this is reasonable where there is curvature). For example, in the massless case with $q=1$ the determinant of the underlying quadratic form is

$$256(k^2k_0^2-4k_0ll_0-l^2l_0^2)(k^2k_0^2+4k_0ll_0-l^2l_0^2)$$

so that there are four 3-surfaces in the four-dimensional metric moduli space where an eigenvalue vanishes (e.g. giving l in terms of k,k_0,l_0).

In short, the two real choices $q=\pm1$ each behave similarly to and deform the main rectangular background case (3.4) for these values in Section 3.2, although the exact eigenvalues and hence the correlation functions depend in a complicated way on the background metric.

4. Quantised metric on a quadrilateral

We now consider quantisation of the general edge-symmetric metric. Again it is convenient to use the Fourier mode expansion as given at the start of Section 3.3 where $k_0, l_0 > 0$ are the average horizontal and vertical square-lengths respectively (the actual horizontal edge weights are negative) and $k = k_1/k_0, l = l_1/l_0$ are the relative fluctuations. Then the Einstein-Hilbert action (2.17) becomes

$$S_g = \sum_{\mathbb{Z}_2\times\mathbb{Z}_2} \mu S = k_0\alpha(k) - l_0\alpha(l); \quad \alpha(k) := \frac{8k^2}{1-k^2} \tag{4.1}$$

in our Lorentzian signature case. This has square-length dimension needing us to divide out by a coupling constant, which we denote G, of square-length dimension.

4.1. Full quantisation.

We functionally integrate over all edge square-lengths with our given Lorentzian signature. Under our change of variables, the measure of integration becomes $\mathrm{d}a_{00}\mathrm{d}a_{01}\mathrm{d}b_{00}\mathrm{d}b_{10} = 4\mathrm{d}k_0\mathrm{d}k_1\mathrm{d}l_0\mathrm{d}l_1 = 4\mathrm{d}k_0\mathrm{d}l_0\mathrm{d}k\mathrm{d}l\, k_0 l_0$ and the partition function becomes $Z = |Z_1|^2$, where

$$\begin{aligned} Z_1 &= 2\int_{-1}^{1}\mathrm{d}k\int_0^L \mathrm{d}k_0 k_0 e^{\frac{\imath}{G}k_0\alpha(k)} = 4G^2\int_0^1 \mathrm{d}k \frac{\mathrm{d}}{\mathrm{d}\alpha}\Big|_{\alpha=\alpha(k)} \frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha} \\ &= 4G^2\int_0^\infty \mathrm{d}\alpha \frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)\end{aligned}$$

for the k_0, k integration while the l_0, l integration gives its complex conjugate. Here we regularised an infinity by limiting the k_0 integral to $0 \le k_0 \le L$ rather than allowing this to be unbounded. We also noted that $\alpha(k)$ is an even function and monotonic in the range $k \in [0,1)$, hence in this range we changed variable to regard $k = \sqrt{\frac{\alpha}{8+\alpha}}$ as a function of $\alpha \in [0,\infty)$. For fixed L the $\int_1^\infty \mathrm{d}\alpha$ part of Z converges (in fact to a bounded oscillatory function of L) but there is a further divergence at $\alpha = 0$. The integrand here is a case of

$$\frac{\mathrm{d}k}{\mathrm{d}\alpha} = \frac{4}{\alpha^{\frac12}(8+\alpha)^{\frac32}}, \quad \frac{\mathrm{d}^m}{\mathrm{d}\alpha^m}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right) = m!\frac{e^{\frac{\imath L}{G}\alpha}e_m^{-\frac{\imath L}{G}\alpha}-1}{(-\alpha)^{m+1}}; \quad e^x_m = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^m}{m!}.$$

Similarly

$$\langle k_0\rangle := \frac{\int_{-1}^1\mathrm{d}k\int_0^L \mathrm{d}k_0 k_0^2 e^{\frac{\imath}{G}k_0\alpha(k)}}{\int_{-1}^1\mathrm{d}k\int_0^L \mathrm{d}k_0 k_0 e^{\frac{\imath}{G}k_0\alpha(k)}} = -\imath G\frac{\int_0^\infty \mathrm{d}\alpha\frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)}{\int_0^\infty \mathrm{d}\alpha\frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)} = -\imath G\lim_{\alpha\to0}\frac{\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)}{\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)} = \frac{2}{3}L.$$

The $\int_1^\infty \mathrm{d}\alpha$ part of the numerator again converges (in fact to a L times a bounded oscillatory function of L) and there is again a divergence at $\alpha = 0$. It follows that the limit of the ratio of the integrals is the limit of the ratio of the integrands at this divergent point. A similar analysis gives in general

$$\langle k_0^m\rangle := \frac{\int_{-1}^1\mathrm{d}k\int_0^L \mathrm{d}k_0 k_0^{m+1} e^{\frac{\imath}{G}k_0\alpha(k)}}{\int_{-1}^1\mathrm{d}k\int_0^L \mathrm{d}k_0 k_0 e^{\frac{\imath}{G}k_0\alpha(k)}} = (-\imath G)^m\frac{\int_0^\infty \mathrm{d}\alpha\frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}^{m+1}}{\mathrm{d}\alpha^{m+1}}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)}{\int_0^\infty \mathrm{d}\alpha\frac{\mathrm{d}k}{\mathrm{d}\alpha}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{1-e^{\frac{\imath L}{G}\alpha}}{\alpha}\right)} = \frac{2}{m+2}L^m.$$

We also have $\langle k_0^m k^n\rangle = 0$ for all $n \ge 1$. For odd n this is clear by parity in the original $\int_{-1}^1 \mathrm{d}k$ but it holds for all positive n because if the ratio of integrands has a limit as

$\alpha \to 0$, an extra factor k in the numerator makes it tend to zero since $k = O(\alpha^{\frac{1}{2}})$. It also does not change that the numerator integral converges as $\alpha \to \infty$ since $k \sim 1$ for large α. In particular, $\langle k \rangle = \langle k^2 \rangle = 0$.

It follows that

$$\langle a_{00} \rangle = \langle a_{01} \rangle = \langle k_0(1 \pm k) \rangle = \frac{2}{3}L$$

and one also has $\langle a_{00} b_{10} \rangle = \langle a_{00} \rangle \langle b_{10} \rangle$ etc since the l_0, l integrals operate independently. We use the same cutoff $0 \leq l_0 < L$. It also follows that

$$\langle a_{00} a_{01} \rangle = \langle a_{00} a_{00} \rangle = \langle a_{01} a_{01} \rangle = \langle k_0^2 (1 \pm k^2) \rangle = \frac{L^2}{2}$$

which implies for example, a relative uncertainty

$$\frac{\Delta a_{00}}{\langle a_{00} \rangle} = \frac{\sqrt{\langle a_{00}^2 \rangle - \langle a_{00} \rangle^2}}{\langle a_{00} \rangle} = \frac{1}{\sqrt{8}} \tag{4.2}$$

for the horizontal edge square-length. Similarly for a_{01} and for b_{00}, b_{10} from the other factor for the vertical theory.

The correlation functions themselves have an infra-red divergence in the same manner as for scalar fields, now appearing as $L \to \infty$ and in principle requiring renormalisation. How to do this in a conventional way is unclear and it may be more appropriate and reasonable (as with the scalar theory) to not renormalise but leave the regulator in place. We can take the operational view that one can cut-off to $L = 3K_0/2$ to land on any desired $\langle a_{00} \rangle = K_0$, then $\langle a_{00}^2 \rangle = \frac{9}{8} K_0^2$ is a calculation for values set at this scale, while the relative $\Delta a / \langle a \rangle$ is independent of this choice of regulator in any case. One might still think of this as some kind of 'field renormalisation' to $\hat{k}_0 = \frac{3K_0}{2L} k_0$ or $\hat{a} = \frac{3K_0}{2L} a$ and similarly for $\hat{b}$. Then $\langle \hat{a}_{00} \rangle = \langle \hat{k}_0 \rangle = K_0$ is any desired value resulting from the bare k_0 cut off at L while $\langle \hat{k}_0^2 (1 \pm k^2) \rangle = \frac{9}{8} K_0^2$ implies the same as (4.2) for the rescaled $\hat{a}_{00}$. However, all we would be doing in practice is replacing k_0 by a new variable $0 \leq \hat{k}_0 \leq \frac{3K_0}{2L} L = 3K_0/2$ so this just amounts to the same as setting $L = 3K_0/2$ in the first place. These is a similar equivalence if one thinks in terms of rescaling the coupling constant G.

One can speculate that the constant relative uncertainty (4.2) suggests some kind of vacuum energy. We also see that a certain amount of geometric structure is necessarily washed out by functional integration in the full quantisation. For example, there is nothing to break the symmetry between a_{00} and a_{01}, just as there was no intrinsic scale for $\langle a_{00} \rangle = \langle a_{01} \rangle$ so it had to be zero or governed by a regulator scale.

4.2. **Quantisation relative to a Lorentzian rectangular background.** By contrast, it also makes sense to quantise about fixed values and indeed to focus on fluctuations from the rectangular case, which we now do in a relative sense. Thus in the Fourier mode decomposition (3.5) of a, b we now fix the average values k_0, l_0 as a background rectangle and only quantise relative fluctuations k, l with action (4.1), where $\alpha(k) = 8(k^2 + k^4 + \ldots)$ is approximately Gaussian as for the scalar field on $\mathbb{Z}_2$ in Section 3.1 (and has its minimum at $k = 0$ as expected) but changes as $|k| \to 1$ in the gravity case. This is not the usual difference fluctuation from a given background, but

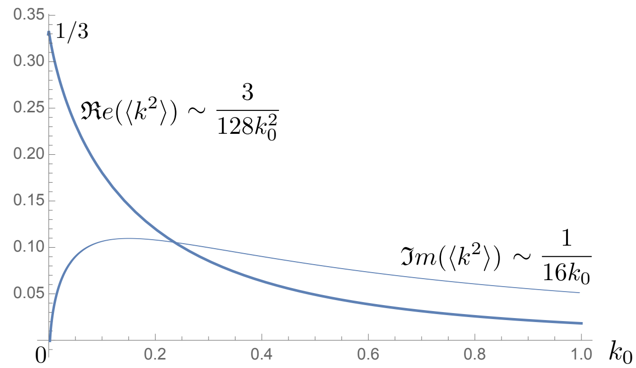

FIGURE 2. Expectation value $\langle k^2\rangle$ for relative quantum metric fluctuations on a background Lorentzian rectangle with sides k_0, l_0 at $G = 1$. Compare with $\langle k^2\rangle = \frac{\imath}{16k_0}$ for the scalar field from (3.1).

fits better with the current computation. In this case,

$$Z(k_0,l_0) = 4k_0l_0\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}l\, e^{\frac{\imath}{G}k_0\alpha(k)-\frac{\imath}{G}l_0\alpha(l)} = 16k_0l_0z(k_0)z(l_0)^*,$$

$$z(k_0) = \int_0^1 e^{\frac{\imath}{G}k_0\alpha(k)}\mathrm{d}k \sim \begin{cases} e^{\frac{\imath\pi}{4}}\sqrt{\frac{\pi G}{32k_0}} & k_0\to\infty\\ 1 & k_0\to 0\end{cases}$$

where we use the latter also for $z(l_0)$. We regard the background rectangle square-lengths $k_0, l_0 > 0$ as coupling constants and the minus sign in the action comes form the Lorentzian signature. This converges and we can similarly compute correlations functions from

$$\langle k^2\rangle = \frac{\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}l e^{\frac{\imath}{G}k_0\alpha(k)-\frac{\imath}{G}l_0\alpha(l)}k^2}{\int_{-1}^{1}\mathrm{d}k\int_{-1}^{1}\mathrm{d}l e^{\frac{\imath}{G}k_0\alpha(k)-\frac{\imath}{G}l_0\alpha(l)}} \sim \frac{G}{16k_0}\imath + \frac{3G^2}{128k_0^2} + \cdots$$

with the indicated asymptotic form at large k_0 shown in Figure 2. Similarly for $\langle l^2\rangle$ with a conjugate answer. From these we have

$$\langle a_{00}\rangle = \langle a_{01}\rangle = \langle k_0(1\pm k)\rangle = k_0$$

$$\langle a_{00}^2\rangle = \langle a_{01}^2\rangle = k_0^2(1+\langle k^2\rangle),\quad \langle a_{00}a_{01}\rangle = k_0^2(1-\langle k^2\rangle)$$

and similarly for b. For the same reasons as before, we also have $\langle a_{00}b_{10}\rangle = k_0l_0$ etc. (and similarly for a, b at any other points). In short, the edge square-lengths a and b behave independently and each is similar to a scalar field, where $\langle k^2\rangle$ asymptotes as $k_0\to\infty$ to the constant imaginary value $\imath G/(16k_0)$ as in (3.1) with $\alpha = 8k_0/G$. We can also compare this with our correlation functions on $\mathbb{Z}_2$ with $\beta = 1$ for a dimensionless scalar field (to match k which is dimensionless). The role of metric square-length a there is now played by k_0 and the role of mass m there is now essentially played by $\sqrt{8/G}$. Although not the same, this gives a feel for the physical picture of the relative quantization of the metric at large k_0. Similarly for large l_0. In both cases 'large' actually means relative to G, so this is equivalent to a small G or weak gravity limit. As we are working in units of $\hbar = c = 1$, this means the background rectangle size should be large compared to the Planck scale.

At the other extreme, the expectation value expanded for small k_0 has a real limit

$$\lim_{k_0\to 0}\langle k^2\rangle = \frac{1}{3}. \tag{4.3}$$

Since k is the amount of fluctuation between the two horizontal metrics edge weights, this again looks like some kind of vacuum energy. It in turn implies a relative edge-length uncertainty in the $k_0 \to 0$ limit of

$$\frac{\Delta a_{00}}{\langle a_{00}\rangle} = \frac{\Delta a_{01}}{\langle a_{01}\rangle} = \sqrt{\langle k^2\rangle} = \frac{1}{\sqrt{3}}$$

similar to our previous result (4.2) in the full quantisation. That this limit should resemble the full theory in this respect indicates that modes with small k_0 dominate the full quantisation. Similarly for $l_0 \to 0$. We also see a bit more structure to the correlation functions with

$$\frac{\langle a_{00}a_{01}\rangle}{\langle a_{00}\rangle\langle a_{01}\rangle} = \frac{2}{3} \tag{4.4}$$

and similarly for the correlation between the two vertical edges in this limit. From the point of view of the present section of quantising fluctuations relative to a fixed background rectangle with sides k_0, l_0, it would be unusual to think geometrically of $k_0, l_0 \to 0$. However, 'small' here actually means relative to G, so this is equivalent to a large G or 'strong gravity' limit. In effect we are taking the background geometry to below the Planck scale which is not very physical but seems to be a useful idealisation.

5. Concluding remarks

We have constructed quantum gravity on a universe of four points with a quadrilateral differential structure and its moduli of quantum Riemannian geometries previously established in [35]. The 'noncommutative' metric and connection here have a plausible if not completely canonical Einstein-Hilbert action, which we now quantised in a functional integral approach. We chose the horizontal edges to be assigned negative values, as a Lorentzian version.

It is a question as to how much we can really learn from such a baby model. Our first answer is that in a usual lattice discretisation we would seek the physical validation by looking to the limit of an infinite number of points. This is certainly possible for the scalar field theory where the case of $\mathbb{Z}$ was covered in [36] rather than the $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ cases in Section 3, but for quantum gravity the finite number of points is exactly what made the theory calculable. If we did have a more complete theory of quantum gravity then we could look back and say that certain features were visible already on $\mathbb{Z}_2 \times \mathbb{Z}_2$, but we are coming at it from the other end and have to take a view as to which features of our model are 'physical' and which are artefacts of the discretisation. This is where noncommutative geometry provides a framework within which geometric concepts are not being approximated or actively discretrised, such as embedded in a continuum; it is just a mathematical fact that differential structures on discrete sets (in a reasonable noncommutative sense) correspond to graphs so just as we might elect to study quantum gravity on this or that continuum manifold of our choice, we are also free to choose to study it as an exact theory on a quadrilateral provided we do so as part of a uniform conception of geometry (as opposed to in an ad-hoc manner). Just as one might choose local (or global) coordinates and Fourier transform so as to have a physical picture with momentum modes, we are also free to do that with our group structure and Cayley graph picture, where left-invariant 1-forms and vector fields provide us a tentative picture of what is going on in more physical terms.

Turning now to our specific results, our first observation is that in our case the edge square-lengths a_{00}, a_{01} on horizontal edges and b_{10}, b_{11} on vertical edges proceeded independently. Thus, we can think of each horizontal edge as a 'point' and a as a real valued scalar field with values on the bottom or top edge (see Figure 1) together with a non-quadratic action $k_0k^2/(1-k^2)$ in terms of the relative fluctuation k and the average horizontal edge square-length k_0. Similarly for the vertical theory based on the variables l, l_0 and the action entering with an opposite sign (so with conjugate results). Because of the latter, the partition function for the full theory was a modulus square of one factor. This horizontal-vertical factorisation is specific to our quadrilateral and indeed we identified a parameter q in the choice of QLC which changes sign if we try to implement a reflection on the diagonal (swapping k_i, l_i) but which is not visible in the Einstein-Hilbert action. On the other hand, the reality of not only the partition function but the correlation functions between metric values was remarkable and could be a more general feature. We also found that both quantum gravity on four points and the massless scalar field theory have an expected IR divergence, regulated by a square-length scale L and mass m respectively. We were able to compute certain correlation functions and found a uniform $1/\sqrt{8}$ relative uncertainty in the metric. It is not clear if this can be interpreted as some kind of vacuum energy but it would appear to be a step in that direction and could be a general feature.

We also learned that rather more structure can be seen by leaving out the zero mode integration in momentum space (the average k_0, l_0 values) and studying the effective action for them given by integrating the fluctuating (non-zero momentum) modes relative to this rectangular background. Here $k, l = 0$ correspond to a rectangle and to extrema of the classical actions, so this is physically reasonable. We found that for large k_0, l_0 or weak G the k, l modes behave like a pair of $\mathbb{Z}_2$ scalar fields of mass $\sqrt{8/G}$, which in our units means of Planck mass order. Meanwhile for small k_0, l_0 or strong G we found that the k, l theory behaved very differently and more like the full quantisation with real and not imaginary correlation functions. These are striking features and it would be interesting to see if some of them are present also in other finite models. For example, taking the background metric to zero is an unexpected but seemingly useful limit that could also be useful in other models.

There are also specific expectations for quantum gravity which could be explored further, even for our baby model of 4 points, possibly with more structure. Aside from the hint of vacuum energy already mentioned, one could explore an expected link between gravity and entropy using the curved background Laplacian of Section 3.3. Cosmological particle creation as done for the discrete lattice line in our related paper [36] is not easily adaptable to four points, but one may still be able to explore issues concerning the vacuum, quantum information and entanglement. At a technical level, it is also possible to look at Einstein's equation in the combined quantisation of both scalar fields and gravity. In such a theory, since the QLC is not uniquely determined by the metric and the scalar Laplacian is sensitive to the ambiguity (which includes the possibility of changing signature), we should really sum or integrate over this freedom in the QLC's. One could look at the theory with a cosmological term (2.18) in the action and one could look at spinor fields in the model, albeit the latter would need more structure. These are some directions for further work even for the baby model of 4 points.

Moreover, the methods of the paper can be applied in principle to other graphs. The quantum Riemannian geometry for the triangle case is solved in [13] and is not too interesting for quantum gravity, but there are many other graphs both finite and infinite (such as the 1-dimensional lattice solved in[36]). Both cases should ultimately be compared with lattice quantum gravity results such as in [1] and with poset models, e.g. [17], although at present the methodologies are very different. In our approach, everything is much easier in the case of a Cayley graph where the differential geometry is parallelisable and allows 'finite-Lie algebra' methods such as we used in Lemma 2.1 to solve for the quantum Riemannian geometry on $\mathbb{Z}_2$, but the definition of a metric and the conditions for a QLC make sense more generally. The key ingredient which was canonical in the Cayley case and which now has to be specified is the choice of Ω^2. Any Ω^1 has a 'maximal prolongation' which provides a candidate Ω^2 but one will typically want to quotient this so as to resemble a classical manifold. There are a lot of possible small graphs but, for example, it might be interesting to solve for quantum gravity on Dynkin diagrams of complex simple Lie algebras as a kind of dual to quantum gravity on the associated Lie group. We also note that the formalism of quantum Riemannian geometry works over any field, opening another front over small fields[38] where QLCs may be more directly computable.

Finally, the full formalism can be applied to other algebras including noncommutative ones, as part of a general framework that is not specific to the graph case. There is, for example, a rich moduli of differential structures and metrics on $M_2(\mathbb{C})$, see [10][13], but their QLCs have so far only been determined for some specific metrics. Here [10] shows, for the specific metric and QLC there and a choice of Clifford structure, how to obtain a Dirac operator on $M_2(\mathbb{C})$ obeying the axioms of a spectral triple in the sense of Connes [14]. The main difference of approach here is that spectral triples are defined as a hermitian operator on the same Hilbert space as on which our 'spacetime' algebra is represented, subject to certain axioms. For a given algebra, however, it is not clear which if any spectral triples have a 'quantum geometric realisation' in the sense of a metric, QLC and Clifford structure as in [10]. Thus for $M_2(\mathbb{C})$ and other matrix algebras, it would be interesting know which of the finite spectral triples in works such as [4, 5, 6] have a geometric realisation. Finite spectral triples on matrix and similar algebras have also been proposed as a way to describe the internal structure of elementary particles, see [15, 44] and references therein, and in that context asking for a geometric realisation as a curved geometry would provide more of a (noncommutative) Kaluza-Klein-like picture. Moreover, the constraint of being geometrically realised would potentially translate into predictions for particle physics. This represents a very different application from quantum gravity but an equally interesting one that should be explored.

One can also solve for quantum metrics and QLCs on other noncommutative algebras, such as enveloping algebras of Lie algebras. For the bicrosssproduct model spacetime, which is in this family, this was done for the two main differential structures in [9, 41] and one is forced to curved metrics. For the spin model or angular momentum algebra $U(su_2)$ as spacetime there is only the flat metric if we take the 4D rotational-invariant calculus[7] (there being no 3D one) and this is relevant to 3D quantum gravity[22], but it remains to consider other quantum differential structures that break rotational invariance and which may admit more interesting quantum Riemannian geometries. For noncommutative continuum models one can also explore the options at leading deformation order in the language of Poisson-Riemannian geometry; for example rotationally invariant quantum Riemannian geometries are then possible on the fuzzy sphere as a quotient of $U(su_2)$ but one must pay the price of nonassociative differentials at the next order in the deformation parameter, see [11, 23] and references therein.

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